ALL LINEAR COISOTROPICS IN $T^*\mathbb{T}^n$ MAY BE FOLIATED BY STANDARD SUBTORI

1. Introduction

A submanifold $\mathcal{C}$ of the symplectic manifold (M, ω) is said to be *coisotropic* if $(T\mathcal{C})^\omega \subset T\mathcal{C}$; i.e., $\mathcal{C}$ is coisotropic if the symplectic orthocomplement to its tangent bundle is a subbundle of the tangent bundle itself. In particular, a coisotropic submanifold of the least allowable dimension is a Lagrangian submanifold.

Every coisotropic submanifold is endowed with a *characteristic foliation*, also referred to as a *null foliation*. In short, the leaves of the characteristic foliation of $\mathcal{C}$ are the integral curves of the Hamiltonian vector fields of its defining functions.

All of our results pertain to linear coisotropic submanifolds in $T^*\mathbb{T}^n$. A d-codimensional linear coisotropic is of the form $\mathcal{C} = \mathbb{T}^n \times \{\mathbf{v}_1 \cdot \xi = \ldots = \mathbf{v}_d \cdot \xi = 0\}$ for $\mathbf{v}_j \in \mathbb{R}^n$ (ξ are symplectically dual to $x \in \mathbb{T}^n$). In this note, we show that

Theorem 1.1. *Let $\mathcal{C} \subset T^*\mathbb{T}^n$ be any linear coisotropic submanifold. Then there exists a linear symplectomorphism Φ of $T^*\mathbb{T}^n$ for which $\Phi(\mathcal{C}) = \widetilde{\mathcal{C}}$, where the leaves of the characteristic foliation of $\widetilde{\mathcal{C}}$, projected down to $\mathbb{T}^n$, are dense in standard subtori.*

(A *standard* k-subtorus is any k-dimensional subtorus of $\mathbb{T}^n$ of the form $\mathbb{T}^k \times \{\texttt{point}\}$.)

2. Models associated to linear coisotropics

Let $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. As coordinates on the symplectic manifold $(T^*\mathbb{T}^n, \omega)$, we take (x, ξ), where $\omega = d\xi \wedge dx$. We study linear coisotropics in $T^*\mathbb{T}^n$, defined as follows:

Definition 2.1. A d-codimensional *linear coisotropic submanifold* $\mathcal{C} \subset T^*\mathbb{T}^n$ has the form

$$\mathcal{C} = \mathcal{C}(\mathbf{v}_1, \ldots, \mathbf{v}_d) = \mathbb{T}^n_x \times \{\mathbf{v}_1 \cdot \xi = \ldots = \mathbf{v}_d \cdot \xi = 0\},$$

where $\{\mathbf{v}_1, \ldots, \mathbf{v}_d\} \subset \mathbb{R}^n$ is linearly independent over $\mathbb{R}$.

A simple linear coisotropic is $\{\xi_1 = \ldots = \xi_d = 0\}$. In fact, locally every coisotropic is of this form [1, Theorem 21.2.4].

We say $\mathcal{C}_1(\mathbf{v}_1, \ldots, \mathbf{v}_d)$ is *symplectically equivalent* to $\mathcal{C}_2(\mathbf{w}_1, \ldots, \mathbf{w}_d)$ if there exists $P \in GL_n(\mathbb{Z})$ such that $P\mathbf{v}_j = \mathbf{w}_j$ for each j. Such a P descends to a diffeomorphism of the torus, which then lifts to a symplectomorphism of $T^*\mathbb{T}^n$ mapping $\mathcal{C}_1$ to $\mathcal{C}_2$.

With the goal of finding 'model' linear coisotropics, we first consider the special case where $\mathbf{v}_j \in \mathbb{Q}^n$. Consider the rational linear coisotropic $\mathcal{C}(\mathbf{v}_1, \ldots, \mathbf{v}_d)$. We may clear denominators and take the $\mathbf{v}_j \in \mathbb{Z}^n$. We identify $\mathcal{C}$ with the matrix $C = (\mathbf{v}_1|\ldots|\mathbf{v}_d)$, and call this a *representation* of $\mathcal{C}$. Since the $\mathbf{v}_j$ are independent, C has rank d. Equivalently, the $\mathbf{v}_j$ are a basis for a rank d sublattice of $\mathbb{Z}^n$.

Proposition 2.2. *Let $\mathcal{D}$ be a PID, and let $A \in M_{n\times d}(\mathcal{D})$ have rank d. Then there exist invertible $P \in M_{n\times n}(\mathcal{D})$, $Q \in M_{d\times d}(\mathcal{D})$ such that $PAQ = (a_1\mathbf{e}_1|\ldots|a_d\mathbf{e}_d)$, where all $a_j \neq 0$.*

This result is adapted from [2, Section 3.7]. Thus, there exists $P \in GL_n(\mathbb{Z})$, $Q \in GL_d(\mathbb{Z})$ for which PCQ has the diagonal form described in the proposition. Note that $(a_1\mathbf{e}_1|\dots|a_d\mathbf{e}_d)$ represents the coisotropic $\{\xi_1 = \dots = \xi_d = 0\}$. The right multiplication by Q merely amounts to a change of basis for the aforementioned rank d sublattice, so C and CQ represent the same coisotropic. Hence, the existence of P tells us that $\mathcal{C}$ is symplectically equivalent to the 'rational model' $\{\xi_1 = \dots = \xi_d = 0\}$; and any two d-codimensional rational linear coisotropics are equivalent, as they are both equivalent to this model.

Next, we allow $\mathbf{v}_j \in \mathbb{R}^n$. Given an arbitrary linear coisotropic $\mathcal{C}(\mathbf{v}_1, \dots, \mathbf{v}_d)$, let $\overline{\mathcal{O}}$ denote the set of leaf closures in the foliation of $\mathcal{C}$, and let $\mathscr{L} \in \overline{\mathcal{O}}$. In the rational case, leaves are *a priori* closed. This implies that $\pi_{\mathbb{T}^n}\mathscr{L}$, the projection of the (closed) leaf onto the torus, is a d-dimensional subtorus. The rational coisotropic is equivalent to $\{\xi_1 = \dots = \xi_d = 0\}$, whose projected leaves are

$$\pi_{\mathbb{T}^n}\mathscr{L} = \{(x_1, \dots, x_d, c_{d+1}, \dots, c_n),\ x_j \in S^1\}.$$

for any choice of the constants c_j.

Definition 2.3. A *standard k subtorus* in $\mathbb{T}^n$ is a subtorus of the form $\mathbb{T}^k \times \{\texttt{point}\} \subset \mathbb{T}^n$.

So the d-codimensional rational model is foliated by standard d subtori. If the $\mathbf{v}_i$ are not assumed to be rational, then $\pi_{\mathbb{T}^n}\mathscr{L}$ are subtori of dimension $k \geq d$. We find 'models' for arbitrary linear coisotropics, in two steps.

Lemma 2.4. *We can express each $\mathbf{v}_i$, $1 \leq i \leq d$, as a linear combination*

$$\mathbf{v}_i = \sum_{j=1}^{k_i} \alpha^i_j \mathbf{w}^i_j$$

such that (1) for each i, $\{\mathbf{w}^i_1, \dots, \mathbf{w}^i_{k_i}\} \subset \mathbb{Z}^n$ is independent, and (2) for each i, the coefficients $\alpha^i_j \in \mathbb{R}$ are $\mathbb{Q}$-independent.

Proof. The proof is the same for each $\mathbf{v}_i$, so fix i and let $\mathbf{v}_i =: \mathbf{v} = (v^1 \cdots v^n)^t$. Consider the subspace $\operatorname{span}_{\mathbb{Q}}\{v^1, \dots, v^n\} \subset \mathbb{R}_{\mathbb{Q}}$, where $\mathbb{R}_{\mathbb{Q}}$ is the space of reals over the rationals. The subspace is finite dimensional; say it has dimension $k \leq n$. Any maximal $\mathbb{Q}$-independent subset of the spanning set $\{v^1, \dots, v^n\}$ is a basis. The argument is analogous for any basis obtained this way, so WLOG take the basis $\{v^1, \dots, v^k\}$. Certainly, for each i in $1 \leq i \leq n$, there exists a unique k-tuple $(c^i_1, \dots, c^i_k) \in \mathbb{Z}^k$ (in $\mathbb{Z}^k$, as opposed to $\mathbb{Q}^k$, after clearing denominators and recalibrating the basis) for which $v^i = \sum_{j=1}^k c^i_j v^j$.

We know that for $1 \leq i \leq k$, $c^i_i = 1$ and $c^i_j = 0$ for $j \neq i$. In general, we have

$$\mathbf{v} = \begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^k c^1_j v^j \\ \vdots \\ \sum_{j=1}^k c^n_j v^j \end{pmatrix} = \sum_{j=1}^k v^j \begin{pmatrix} c^1_j \\ \vdots \\ c^n_j \end{pmatrix}.$$

Let $\alpha_j := v^j$; since the v^j, $1 \leq j \leq k$, are $\mathbb{Q}$-independent (being a basis), this choice of α_j fulfills condition (2). Also, let $\mathbf{w}_j := \left(c^1_j \cdots c^n_j\right)^t \in \mathbb{Z}^n$. It is clear that $\{\mathbf{w}_j\}_{j=1}^k$ is linearly independent: if we were to write the $n \times k$ matrix $(\mathbf{w}_1|\dots|\mathbf{w}_k)$, the $k \times k$ minor matrix formed with the first k rows would just be the identity matrix. So condition (1) is satisfied. $\square$

The decomposition in the lemma is not unique. For example, suppose all entries in $\mathbf{v}_j$ are $\mathbb{Q}$-independent. Then for any $A \in GL_n(\mathbb{Z})$, we may write $\mathbf{v}_j$ as a linear combination of the columns of A, with the entries of $A^{-1}\mathbf{v}_j$ being the coefficients.

Lemma 2.5. *Assume $\pi_{\mathbb{T}^n}\mathscr{L}$ is a k-dimensional subtorus. Then there exists independent $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\} \subset \mathbb{Z}^n$ and for each i, $1 \le i \le d$, there exist $\beta^i_m \in \mathbb{R}$ $(1 \le m \le k)$ for which*

$$\mathbf{v}_i = \sum_{m=1}^{k} \beta^i_m \mathbf{u}_m. \tag{2.1}$$

For fixed i, $1 \le i \le d$, the coefficients β^i_m may not be $\mathbb{Q}$-independent: see Example 2.7.

Proof. After decomposing each $\mathbf{v}_i$, $1 \le i \le d$, as in Lemma 2.4, we next pool all the $\mathbf{w}^i_j$ together to obtain the set $S = \{\mathbf{w}^i_j\}_{1 \le i \le d, 1 \le j \le k_i}$ consisting of as many as $\sum_{i=1}^d k_i$ distinct vectors. Then we trim S by finding *any* maximal linearly independent subset S'; S' must contain exactly k elements, because $\pi_{\mathbb{T}^n}\mathscr{L}$ was assumed to be k-dimensional. We relabel elements and write $S' = \{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$.

Then S' is a basis for $\mathrm{span}_{\mathbb{R}}(S)$. For each i, $\mathbf{v}_i \in \mathrm{span}_{\mathbb{R}}\{\mathbf{w}^i_1, \ldots, \mathbf{w}^i_{k_i}\} \subset \mathrm{span}_{\mathbb{R}}(S)$. Thus, each $\mathbf{v}_i$ can be written as a linear combination of the elements of S'. □

This leads to our main result.

Theorem 2.6. *Let $\mathcal{C} = \mathcal{C}(\mathbf{v}_1, \ldots, \mathbf{v}_d)$ for $\mathbf{v}_1, \ldots, \mathbf{v}_d \in \mathbb{R}^n$. Suppose that $\pi_{\mathbb{T}^n}\mathscr{L}$ is k-dimensional. Then there exists a linear symplectomorphism $\Phi : T^*\mathbb{T}^n \to T^*\mathbb{T}^n$ such that $\Phi(\mathcal{C}) = \widetilde{\mathcal{C}}$, where the leaves of the null foliation of $\widetilde{\mathcal{C}}$, projected down to the torus, are dense in the standard k-dimensional subtori of $\mathbb{T}^n$. (That is, $\widetilde{\mathcal{C}} = \mathcal{C}(\tilde{\mathbf{v}}_1, \ldots, \tilde{\mathbf{v}}_d)$, where $\tilde{\mathbf{v}}_i \in \mathbb{R}^k \times \{0\} \subset \mathbb{R}^n$.)*

Proof. According to Lemma 2.5, for each i, $1 \le i \le d$, we may write $\mathbf{v}_i = \sum_{m=1}^k \beta^i_m \mathbf{u}_m$, for real coefficients β^i_m. Then we have

$$\mathcal{C} = \left\{ \sum_{m=1}^{k} \beta^1_m \mathbf{u}_m \cdot \xi = \ldots = \sum_{m=1}^{k} \beta^d_m \mathbf{u}_m \cdot \xi = 0 \right\}.$$

We then form the rank k matrix $(\mathbf{u}_1 | \ldots | \mathbf{u}_k)$, and (possibly after changing bases for the sublattice of $\mathbb{Z}^n$ generated by $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$) convert it to the normal form $(a_1\mathbf{e}_1 | \ldots | a_k\mathbf{e}_k)$, from which arises the coisotropic

$$\widetilde{\mathcal{C}} = (\tilde{\mathbf{v}}_1, \ldots, \tilde{\mathbf{v}}_d), \quad \tilde{\mathbf{v}}_i := \sum_{m=1}^{k} \beta^i_m a_m \mathbf{e}_m.$$

Since $(\mathbf{u}_1 | \ldots | \mathbf{u}_k)$ is converted to $(a_1\mathbf{e}_1 | \ldots | a_k\mathbf{e}_k)$, there exists $P \in GL_n(\mathbb{Z})$ such that

$$(P\mathbf{u}_1 | \ldots | P\mathbf{u}_k) = (a_1\mathbf{e}_1 | \ldots | a_k\mathbf{e}_k).$$

Define $\Phi : (x, \xi) \longmapsto \left(Px, (P^{-1})^t \xi\right)$, a linear symplectomorphism of $T^*\mathbb{T}^n$. It remains to show that Φ maps $\mathcal{C}$ to $\widetilde{\mathcal{C}}$. Suppose $(x, \xi) \in \mathcal{C}$. Then

$$\tilde{\mathbf{v}}_i \cdot \left(P^{-1}\right)^t \xi = \sum_{m=1}^{k} \beta^i_m a_m \mathbf{e}_m \cdot \left(P^{-1}\right)^t \xi = \sum_{m=1}^{k} \beta^i_m P^{-1}(a_m \mathbf{e}_m) \cdot \xi = \sum_{m=1}^{k} \beta^i_m \mathbf{u}_m \cdot \xi = \mathbf{v}_i \cdot \xi = 0,$$

as required.

Finally, since the last $n-k$ entries of each $\tilde{\mathbf{v}}_i$ are zero, the projected leaf closures of $\widetilde{\mathcal{C}}$ are the standard k-subtori. ☐

Example 2.7. Consider the coisotropic $\mathcal{C}(\mathbf{v}_1, \mathbf{v}_2)$ with $\mathbf{v}_1 = (1\ 0\ \pi\ \pi)^t$, $\mathbf{v}_2 = (e\ 1\ e\ e)^t$. By the method in the proof of the theorem, we may produce

$$\widetilde{\mathcal{C}}(\tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_2), \quad \tilde{\mathbf{v}}_1 = (1\ 0\ \pi\ 0)^t, \quad \tilde{\mathbf{v}}_1 = (e\ 1\ e\ 0)^t,$$

which is symplectically equivalent to $\mathcal{C}$, and whose foliation is by *standard* 3-subtori $\pi_{\mathbb{T}^4}\mathscr{L} = \{(x_1, x_2, x_3, c_4)\}$.

Let $\mathcal{C} \subset T^*\mathbb{T}^n$ be any d-codimensional linear coisotropic submanifold, so that $\mathcal{C}$ is foliated, in $\mathbb{T}^n$, by (possibly nonstandard) k-dimensional subtori ($d \leq k$).

Definition 2.8. Let $\mathcal{M}$ be a linear coisotropic that is symplectically equivalent to $\mathcal{C}$. If $\mathcal{M}$ is of the form $\{\mathbf{u}_1 \cdot \xi = \ldots = \mathbf{u}_d \cdot \xi = 0\}$, where $\mathbf{u}_j \in \mathbb{R}^k \times \{0\}$, then we call $\mathcal{M}$ a *model coisotropic associated with* $\mathcal{C}$.

Each leaf in the foliation of any model coisotropic associated with $\mathcal{C}$, projected to the base, is dense in a standard k-dimensional subtorus $\mathbb{T}^k \times \{\texttt{point}\}$. Theorem 2.6 tells us that all linear coisotropics have an associated model. Careful inspection of the example shows that models associated with a given linear coisotropic are not unique.

References

[1] Lars Hörmander, *The analysis of linear partial differential operators III*, Pseudo-Differential Operators, Springer Verlag, Classics in Mathematics, Berlin, 1985.

[2] Nathan Jacobson, *Basic algebra I*, W.H. Freeman and Company, second edition, New York, 1985.